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The Essential Component of the Solution Set for Vector Equilibrium Problems*

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Abstract. In the paper, by using Ky Fan's section theorem, we obtain an existence theorem for vector equilibrium problems. Motivated by the ideas of Kinoshita and McLennan, we introduce the concept of the essential component of the solution set for vector equilibrium problems, and we prove that there exists at least one essential component of the solution set for every vector equilibrium problem satisfying some conditions.

Key words: C- function, essential component, vector equilibrium problem

1. Introduction and Preliminaries

Ansari [1] studied the vector equilibrium problems, and obtained some existence theorems. In the paper, by using Ky Fan's section theorem, we obtain an existence theorem for vector equilibrium problems. McLennan [8] had a comprehensive study on the stability problems of fixed points. Motivated by the ideas of Kinoshita [6] and McLennan [8], we introduce the concept of the essential component of the solution set for vector equilibrium problems, and prove that there exists at least one essential component of the solution set for every vector equilibrium problem satisfying some conditions.

For the sake of convenience, we first introduce some definitions and notations.

Let Y be a real topological vector space. A nonempty subset C in Y is called a pointed cone if C is a cone and $C \cap (-C) = \{0\}$.

Let H and Y be two topological vector spaces and X a nonempty and convex subset of H. Let $f\colon X \times X \to Y$ with $f(x,x) = 0,\ \forall x \in X$ and $\{C(x)\colon x \in X\}$ be a family of closed, pointed and convex cones in Y with int $C(x) \neq \emptyset$, where int $C(x)$ denotes the interior of the set $C(x)$. The *Vector Equilibrium Problem* (for short, VEP) is:

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Find $y^* \in X$, such that

$$f(y^*, x) \notin \text{ int } C(y^*), \quad \forall x \in X.$$

Let T be a set-valued map from a Hausdorff topological space H to another Y, we say that T is upper semicontinuous (in short, u.s.c) at $x_0 \in H$ if for any neighborhood $N(T(x_0))$ of $T(x_0)$, there exists an open neighborhood $O(x_0)$ of x_0 such that

$$\forall x \in O(x_0), \quad T(x) \subset N(T(x_0)).$$

We say that T is upper semicontinuous if T is u.s.c. at every point $x \in H$.

If Y is a metric space, H is a Hausdorff topological space, 2^Y denotes all the subsets of Y, and the set-valued map $T: H \to 2^Y$ is compact for each $x \in H$, then, by Corollary 4.2.3 of [7], T is continuous at $x_0 \in H$ if and only if for each $\varepsilon > 0$, there exists an open neighborhood $O(x_0)$ of x_0 such that $h(T(x_0), T(x)) < \varepsilon$ for all $x \in O(x_0)$, where h is the Hausdorff metric defined on Y.

DEFINITION 1.1. Let X be a nonempty and convex subset of a topological vector space H and let Y be a topological vector space with a closed and convex cone C such that int $C \neq \emptyset$. A mapping $p: X \to Y$ is called C-function, iff $\forall x, y \in X$, $\forall \lambda \in [0, 1]$,

$$p(\lambda x + (1-\lambda)y) - \lambda p(x) - (1-\lambda)p(y) \in C.$$

DEFINITION 1.2. A subset $Q \subset Y$ is residual if it contains a countable intersection of open dense subsets.

It is easy to see that any residual subset in Baire space is dense.

The following statement is the so-called Fort Theorem (to see [5]).

LEMMA 1.1 ([5]). *If X is a metric space, Y is a Baire space, and $G: Y \to 2^X$ is upper semicontinuous with compact values, then, there exists a dense residual set Q in Y such that G is lower semicontinuous on Q.*

Let (E, C) be an ordered topological vector space with a closed, pointed and convex cone C with int $C \neq \emptyset$. $y_1, y_2 \in E$, $y_1 \leq_C y_2 \Longleftrightarrow y_2 - y_1 \in C$.

LEMMA 1.2 ([2]). *Let (E, C) be an ordered topological vector space with a closed, pointed and convex cone C with int $C \neq \emptyset$. Then, $\forall y, z \in E$, we have*

$$y - z \in C \text{ and } y \notin \text{ int } C \text{ imply that } z \notin \text{ int } C.$$

LEMMA 1.3 (Ky Fan's Section Theorem [4]). *Let X be a nonempty, compact and convex set of a Hausdorff topological vector space H, $A \subset X \times X$ having the following properties:*

(1) *for any $x \in X$, $(x, x) \in A$;*
(2) *for any $x \in X$, $A_x = \{y \in X: (x, y) \in A\}$ is closed in X;*
(3) *for any $y \in X$, $A^y = \{x \in X: (x, y) \notin A\}$ is convex.*

Then there exists $y^* \in X$, such that $X \times \{y^*\} \subset A$.

2. Existence Theorem

THEOREM 2.1. *Let X be a nonempty, compact and convex subset of a Hausdorff topological vector space H, E a Hausdorff topological vector space, $f: X \times X \to E$, $C: X \to 2^E$, and suppose that the following conditions are satisfied:*

(i) *for any $x \in X$, $C(x)$ is a closed, pointed and convex cone with $int\ C(x) \neq \emptyset$;*
(ii) *for any $x \in X$, $f(\cdot, x)$ is continuous on X;*
(iii) *for any $y \in X$, the set $\{x \in X: f(y, x) \in int\ C(y)\}$ is convex;*
(iv) *for any $x \in X$, $f(x, x) = 0$;*
(v) *the map $W: X \to 2^E$ is upper semicontinuous, where $W(x) = E \backslash (int\ C(x))$.*

Then, there exists $y^* \in X$ such that $f(y^*, x) \notin int\ C(y^*)$, $\forall x \in X$.

Proof. Let $A = \{(x, y) \in X \times X: f(y, x) \notin \text{int } C(y)\}$.

For any $x \in X$, by (i), $f(x, x) = 0 \notin \text{int } C(x)$, then, $(x, x) \in A$, and $A_x = \{y \in X: (x, y) \in A\} = \{y \in X: f(y, x) \notin \text{int } C(y)\} = \{y \in X: f(y, x) \in W(y)\}$.

Suppose $y_\alpha \in A_x$ with $y_\alpha \to y_0 \in X$. Then, $f(y_\alpha, x) \in W(y_\alpha)$, by (ii), $f(y_\alpha, x) \to f(y_0, x)$, by (v), $f(y_0, x) \in W(y_0)$, i.e., $f(y_0, x) \notin \text{int } C(y_0)$, $y_0 \in A_x$, then, A_x is closed.

By (iii), for any $y \in X$, $A^y = \{x \in X: (x, y) \notin A\} = \{x \in X: f(y, x) \in \text{int } C(y)\}$ is convex. By Lemma 1.3, there exists $y^* \in X$, such that $X \times \{y^*\} \subset A$, then

$$f(y^*, x) \notin \text{int } C(y^*), \quad \forall x \in X. \qquad \square$$

By the definition of C-function, it is easy to prove the following lemmas, the proofs are omitted.

LEMMA 2.2. *Let X be a convex subset of a Hausdorff topological vector space H, (E, C) an ordered topological vector space with a closed, pointed and convex cone C with $int\ C \neq \emptyset$. If $f: X \to E$ is C-function, then, the set $\{x \in X: f(x) \in int\ C\}$ is convex.*

LEMMA 2.3. *Let X be a convex subset of a Hausdorff topological vector space H, (E, C) an ordered topological vector space with a closed, pointed and convex cone C with int $C \neq \emptyset$. If the mappings $f: X \to E$ and $g: X \to E$ are C-functions, then, for any $t \in [0, 1]$, the map $tf + (1-t)g: X \to E$ is also a C-function, where $(tf + (1-t)g)(x) = tf(x) + (1-t)g(x)$ for any $x \in X$.*

3. The Essential Component

Throughout this section, X denotes a nonempty compact convex subset of a Banach space. Let (E, C) be an ordered Banach space with a closed, pointed and convex cone C with int $C \neq \emptyset$.

Let M be the collection of all maps $f: X \times X \to E$ satisfying the following conditions:

(1) for any $x \in X$, $f(x, x) = 0$;
(2) for any $x \in X$, $f(\cdot, x)$ is continuous on X;
(3) for any $y \in X$, the map $f(y, \cdot)$ is a C-function;

For any $f, g \in M$, we define

$$\rho(f, g) = \sup_{(x,y)\in X\times X} d(f(x, y), g(x, y)),$$

where d is the metric which is induced by the norm $\|\cdot\|$ on E. Clearly, (M, ρ) is a complete metric space.

For any $f \in M$, by Theorem 2.1 and Lemma 2.2, there exists $x^* \in X$ such that

$$f(x^*, x) \notin \text{ int } C, \quad \text{for any } x \in X.$$

Denote by $F(f)$ the set of all the solutions of *The Vector Equilibrium Problem f*, i.e.,

$$F(f) = \{x \in X: \forall y \in X, \ f(x, y) \notin \text{ int } C\}.$$

Thus, $f \to F(f)$ indeed defines a solution mapping $F: M \to 2^X$ and we have:

THEOREM 3.1. *The solution mapping $F: M \to 2^X$ is upper semicontinuous mapping with compact values.*

Proof For any $f \in M$, we will prove that $F(f) = \{x \in X: \forall y \in X, f(x, y) \notin$ int $C\}$ is compact.

Let $x_n \in F(f)$ with $x_n \to x^* \in X$ $(n \to \infty)$.

Suppose $x^* \notin F(f)$. Then, there exists $y_0 \in X$ such that $f(x^*, y_0) \in$ int C. Since $f(\cdot, y_0)$ is continuous at x^*, $f(x_n, y_0) \to f(x^*, y_0)$, and int$C$ is open, there exists n_0 such that $\forall n > n_0$, $f(x_n, y_0) \in$ int C which contradicts the fact that $x_n \in F(f)$. Then, $x^* \in F(f)$, $F(f)$ is closed and hence compact.

Because X is compact, we only need to prove that the Graph $F = \{(f, x) \in M \times X : x \in F(f)\}$ is closed.

Let $(f_n, x_n) \in$ Graph F with $(f_n, x_n) \to (f^*, x^*) \in M \times X$ $(n \to \infty)$, then $x_n \in F(f_n)$, i.e., $f_n(x_n, y) \notin$ int C, for any $y \in X$.

Suppose $x^* \notin F(f^*)$, then, there exists $y \in X$ such that $f^*(x^*, y) \in$ int C, since $(f_n, x_n) \to (f^*, x^*)$ and $f^*(\cdot, y)$ is continuous at x^*, $f_n(x_n, y) \to f^*(x_n, y) \to f^*(x^*, y)$.

Since int C is open, there exists n_1 such that $\forall n > n_1$, $f_n(x_n, y) \in$ int C which contradicts the fact that $x_n \in F(f_n)$. Then, $x^* \in F(f^*)$, the Graph$F$ must be closed and the map F is upper semicontinuous with compact values. □

By Lemma 1.1 and Theorem 3.1, we have

THEOREM 3.2. *There exists a dense residual subset Q of M such that for any $f \in Q$, F is lower semicontinuous at f, and hence F is continuous at f, i.e., $F(f)$ is stable ($F(f)$ is called stable if, F is continuous at f).*

In the sense of Baire category, the solution sets of most of the vector equilibrium problems are stable. However, there exist vector equilibrium problems, their solution sets are not stable.

EXAMPLE 3.1. Let $X = [-1, 1]$, $E = (-\infty, +\infty)$, $C = [0, +\infty)$, $f : X \times X \to R$, for any $x, y \in X$, $f(x, y) \equiv 0$, then, $F(f) = X$.

$\forall \varepsilon > 0$, take $f_\varepsilon(x, y) = \frac{\varepsilon}{2}(x^2 - y^2)$, it is easy to see that $f_\varepsilon \in M$ and $\rho(f, f_\varepsilon) \to 0$ $(\varepsilon \to 0)$, however, $F(f_\varepsilon) = \{0\}$, i.e., $F(f_\varepsilon) \not\to F(f)$ $(\varepsilon \to 0)$, that is, the map F is not continuous at f and hence $F(f)$ is not stable.

We introduce the notion of the essential component of the solution set for vector equilibrium problems, and we prove that for any $f \in M$, there exists at least one essential component of the solution set $F(f)$.

DEFINITION 3.1. For $f \in M$, $x \in F(f)$ is called an essential solution point of f, if for any open neighborhood $O(x)$ of x in X, there exists $\delta > 0$ such that $O(x) \cap F(f') \neq \emptyset$ for any $f' \in M$ with $\rho(f, f') < \delta$.

For each $f \in M$, the component of a point $x \in F(f)$ is the union of all connected subsets of $F(f)$ which contain the point x, see pp. 356 in [3]. Components are connected closed subsets of $F(f)$ and are also connected and compact. It is easy to see that the components of two distinct points of $F(f)$ either coincide or are disjoint, so that all components constitute a decomposition of $F(f)$ into connected pairwise disjoint compact subsets, i.e.,

$$F(f) = \bigcup_{\alpha \in \Lambda} C_\alpha(f),$$

where Λ is an index set; for any $\alpha \in \Lambda$, $C_\alpha(f)$ is nonempty, connected and compact, furthermore, for any α, $\beta \in \Lambda (\alpha \neq \beta)$, $C_\alpha(f) \cap C_\beta(f) = \emptyset$.

DEFINITION 3.2. For $f \in M$, $C_\alpha(f)$ is called an essential component if for each open set O containing $C_\alpha(f)$, there exists $\delta > 0$ such that for any $g \in M$ with $\rho(f, g) < \delta$, $F(g) \cap O \neq \emptyset$.

Remark 3.1. For $f \in M$, if $x \in F(f)$ is an essential solution point, then, the component which contains the point x is an essential component.

Remark 3.2. In Example 3.1, there exists no essential solution point of $F(f)$.

THEOREM 3.3. *For any $f \in M$, there exists at least one essential component of $F(f)$.*

Proof. For $f \in M$, suppose that $F(f)$ is decomposed as follows:

$$F(f) = \bigcup_{\alpha \in \Lambda} C_\alpha(f),$$

where Λ is an index set; for any $\alpha \in \Lambda$, $C_\alpha(f)$ is nonempty, connected and compact, furthermore, for any α, $\beta \in \Lambda (\alpha \neq \beta)$, $C_\alpha(f) \cap C_\beta(f) = \emptyset$.

We will prove that there exists at least one essential component. Otherwise, for every $\alpha \in \Lambda$, there exists an open set $O_\alpha \supset C_\alpha$ such that for any $\delta > 0$, there exists $g_\alpha \in M$ with $\rho(f, g_\alpha) < \delta$, $F(g_\alpha) \cap O_\alpha = \emptyset$.

Following the proof of Lemma 1 in [6], because $F(f)$ is compact, there exist two open coverings $\{V_i\}_{i=1}^n$ and $\{W_i\}_{i=1}^n$ which satisfy the following conditions:

(1) $\overline{W}_i \subset V_i$, where $\overline{W}_i$ is the closure of W_i;
(2) $V_i \cap V_j = \emptyset$, $i \neq j$;
(3) V_i contains at least one C_{α_i} with $O_{\alpha_i} \supset V_{\alpha_i}$.

By Theorem 3.1, F is upper semicontinuous at f, since $\cup_{i=1}^n W_i \supset F(f)$ and $\cup_{i=1}^n W_i$ is open, there exists $\delta^* > 0$, for any $f' \in M$ with $\rho(f, f') < \delta^*$, $\cup_{i=1}^n W_i \supset F(f')$.

Thus, there exist $f_{\alpha_i} \in M$ with $\rho(f, f_{\alpha_i}) < \delta^*$ such that $F(f_{\alpha_i}) \cap O_{\alpha_i} = \emptyset$.
We define a map $f^*: X \times X \to E$ as follows:

$$f^*(x, y) = \begin{cases} f(x, y), & \text{if } x \in X \setminus \bigcup_{i=1}^n V_i,\ y \in X; \\ g_{\alpha_i}(x, y), & \text{if } x \in \overline{W}_i,\ y \in X; \\ \lambda_i(x) f(x, y) + \mu_i(x) f_{\alpha_i}(x, y), & \text{if } x \in V_i \setminus \overline{W}_i,\ y \in X, \end{cases}$$

where

$$\lambda_i(x) = \frac{d(x, \overline{W}_i)}{d(x, \overline{W}_i) + d(x, X \setminus \bigcup_{i=1}^n V_i)},$$

$$\mu_i(x) = \frac{d(x, X \setminus \bigcup_{i=1}^n V_i)}{d(x, \overline{W}_i) + d(x, X \setminus \bigcup_{i=1}^n V_i)}.$$

Note that $\lambda_i(x)$ and $\mu_i(x)$ are continuous, $\lambda_i(x) \geqslant 0$, $\mu_i(x) \geqslant 0$, and $\lambda_i(x) + \mu_i(x) = 1$.

By Lemma 2.3, it is easy to see that $f^* \in M$ and $\rho(f, f^*) < \delta$, then, $F(f^*) \neq \emptyset$. Thus, $F(f^*) \subset \cup_{i=1}^n W_i$.

For any $x_0 \in F(f^*)$, there exists i_0 such that $x_0 \in W_{i_0} \subset \overline{W}_{i_0}$, i.e., $x_0 \in F(f_{\alpha_{i_0}})$ which contradicts the fact that $O_{\alpha_{i_0}} \supset V_{i_0} \supset \overline{W}_{i_0}$, $F(f_{\alpha_{i_0}}) \cap O_{\alpha_{i_0}} = \emptyset$. Then, there exists at least one essential component of $F(f)$ and the proof is complete. □

Remark 3.3. In example 3.1, the essential component of $F(f)$ is $[-1, 1]$. In fact, for any open set O which contain $[-1, 1]$, take δ: $0 < \delta < 1$, for any $g \in M$ with $\rho(f, g) < \delta$, we have $F(g) \subset [-1, 1]$, then $F(g) \cap O \neq \emptyset$.

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